

1. The orientation of a  $\xi - \eta - \zeta$  frame with respect to a nonmoving  $x - y - z$  frame is described by a set of Euler parameters that can be determined from the following angle and unit vector:

$$\phi = 60^\circ \quad \mathbf{u} = \begin{pmatrix} 0.4 \\ 0.6 \\ -0.7 \end{pmatrix}$$

- (a) Compute the Euler parameters, construct matrices  $\mathbf{G}$  and  $\mathbf{L}$ , and  $\mathbf{A}$ .

**Solution.**

$$e_0 = \cos \frac{60^\circ}{2} = \frac{\sqrt{3}}{2}$$

$$\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \mathbf{u} \sin \frac{\phi}{2} = \begin{pmatrix} 0.2 \\ 0.3 \\ -0.35 \end{pmatrix}$$

From these results, we can construct the all of the desired matrices.

$$\mathbf{G} = \begin{pmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{pmatrix} = \begin{pmatrix} -0.2000 & 0.8660 & 0.3500 & 0.3000 \\ -0.3000 & -0.3500 & 0.8660 & -0.2000 \\ 0.3500 & -0.3000 & 0.2000 & 0.8660 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{pmatrix} = \begin{pmatrix} -0.2000 & 0.8660 & -0.3500 & -0.3000 \\ -0.3000 & 0.3500 & 0.8660 & 0.2000 \\ 0.3500 & 0.3000 & -0.2000 & 0.8660 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{GL}^T = \begin{pmatrix} 0.5775 & 0.7262 & 0.3796 \\ -0.4862 & 0.6775 & -0.5564 \\ -0.6596 & 0.1364 & 0.7425 \end{pmatrix}$$

- (b) Test  $\mathbf{A}^T \mathbf{A}$ ,  $\mathbf{p}^T \mathbf{p} - 1$ , and also  $\det(\mathbf{A})$  for orthogonality.

**Solution.** Calculate all of the identities,

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} 1.0049 & 0.0000 & 0.0000 \\ 0.0000 & 1.0049 & 0.0000 \\ 0.0000 & 0.0000 & 1.0049 \end{pmatrix}$$

which is close enough given the rounding in the matrix  $\mathbf{A}$ .

$$\mathbf{p}^T \mathbf{p} - 1 = 0.0025$$

$$\det(\mathbf{A}) = 1.0074$$

(c) If  $|\det(\mathbf{A})| > 10^{-6}$ , then normalize vector  $\mathbf{u}$ .

**Solution.** I assume that the condition should be  $|\det(\mathbf{A}) - 1| > 10^{-6}$  given the above result. Normalize  $\mathbf{u}$ .

$$\mathbf{u} = \begin{pmatrix} 0.3980 \\ 0.5970 \\ -0.6965 \end{pmatrix}$$

(d) Repeat parts (a) and (b). Observe the changes in  $\mathbf{A}^T \mathbf{A}$  and  $\mathbf{p}^T \mathbf{p} - 1$ .

**Solution.**

$$e_0 = \cos \frac{60^\circ}{2} = \frac{\sqrt{3}}{2}$$

$$\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \mathbf{u} \sin \frac{\phi}{2} = \begin{pmatrix} 0.1990 \\ 0.2985 \\ -0.3483 \end{pmatrix}$$

From these results, we can construct the all of the desired matrices.

$$\mathbf{G} = \begin{pmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{pmatrix} = \begin{pmatrix} -0.1990 & 0.8660 & 0.3483 & 0.2985 \\ -0.2985 & -0.3483 & 0.8660 & -0.1990 \\ 0.3483 & -0.2985 & 0.1990 & 0.8660 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{pmatrix} = \begin{pmatrix} -0.1990 & 0.8660 & -0.3483 & -0.2985 \\ -0.2985 & 0.3483 & 0.8660 & 0.1990 \\ 0.3483 & 0.2985 & -0.1990 & 0.8660 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{G}\mathbf{L}^T = \begin{pmatrix} 0.5791 & 0.7221 & 0.3784 \\ -0.4845 & 0.6781 & -0.5526 \\ -0.6556 & 0.1367 & 0.7426 \end{pmatrix}$$

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} 0.9999 & 0.0000 & 0.0000 \\ 0.0000 & 0.9999 & 0.0000 \\ 0.0000 & 0.0000 & 0.9999 \end{pmatrix}$$

which is close enough given the rounding in the matrix  $\mathbf{A}$ .

$$\mathbf{p}^T \mathbf{p} - 1 = -2.7860 \times 10^{-5}$$

$$|\det(\mathbf{A}) - 1| = 8.3578 \times 10^{-5}$$

The values for the quantities are much closer to what is expected.

2. Use the normalized (corrected) Euler parameters from Problem 1. Assume the time derivative of Euler parameters is given as:

$$\dot{\mathbf{p}} = (-0.1418 \quad -0.0990 \quad 0.1451 \quad -0.2961)^T$$

(a) Test  $\mathbf{p}^T \dot{\mathbf{p}}$  for orthogonality; if not orthogonal, normalize  $\dot{\mathbf{p}}$  according to the following:

Denote the non-orthogonal  $\dot{\mathbf{p}}$  as  $\dot{\mathbf{p}}^*$ . Compute the normalized  $\dot{\mathbf{p}}$  as

$$\dot{\mathbf{p}} = \dot{\mathbf{p}}^* - (\mathbf{p}^T \dot{\mathbf{p}}^*) \mathbf{p}$$

**Solution.**

$$\mathbf{p}^T \dot{\mathbf{p}} = 0.0039.$$

This value is larger than expected, so normalize  $\dot{\mathbf{p}}$ .

$$\dot{\mathbf{p}} = (-0.1452 \quad -0.0998 \quad 0.1439 \quad -0.2947)^T$$

(b) Repeat part (a). Observe changes in  $\mathbf{p}^T \dot{\mathbf{p}}$ .

**Solution.**

$$\mathbf{p}^T \dot{\mathbf{p}} = 1.0988 \times 10^{-7}$$

This value is much closer to the expected value of 0.

(c) Compute components of the angular velocity vector in  $\xi - \eta - \zeta$  frame and in  $x - y - z$  frame.

**Solution.** In the  $\xi - \eta - \zeta$  frame, we have

$$\omega' = 2\mathbf{G}\dot{\mathbf{p}} = \begin{pmatrix} -0.1907 \\ 0.5228 \\ -0.4948 \end{pmatrix}$$

In the  $x - y - z$  frame, we have

$$\omega = 2\mathbf{L}\dot{\mathbf{p}} = \begin{pmatrix} -0.0393 \\ 0.1492 \\ -0.7285 \end{pmatrix}$$

3. Point  $P$  is located from the origin of a coordinate system by a vector  $\vec{s}$  having components

$$(4, 3, -6)^T$$

This vector is subjected to a rotation of  $50^\circ$  positive about an axis passing through the origin and a point  $A$  having coordinates  $(-4, 3, 2)^T$ . Find the components of vector  $\vec{s}$  in its new orientation.

**Solution.** First, normalize the vector  $\mathbf{u}$  passing through the origin and point  $A$ .

$$\mathbf{u} = \begin{pmatrix} -0.7428 \\ 0.5571 \\ 0.3714 \end{pmatrix}$$

We can derive the Euler parameters from this unit vector and the rotation angle given,

$$e_0 = \cos \frac{50^\circ}{2} = 0.9063$$

$$\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \mathbf{u} \sin \frac{50^\circ}{2} = \begin{pmatrix} -0.3139 \\ 0.2354 \\ 0.1570 \end{pmatrix}$$

Compute  $\mathbf{A}$  from  $\mathbf{G}$  and  $\mathbf{L}$ ,

$$\mathbf{A} = \mathbf{GL}^T = \begin{pmatrix} 0.8399 & -0.4324 & 0.3281 \\ 0.1368 & 0.7536 & 0.6429 \\ -0.5253 & -0.4951 & 0.6921 \end{pmatrix}$$

Now rotate  $\mathbf{s}$  into the new frame,

$$\mathbf{s}' = \mathbf{As} = \begin{pmatrix} 0.0936 \\ -1.0493 \\ -7.7387 \end{pmatrix}$$